Trivial tasks in 3D

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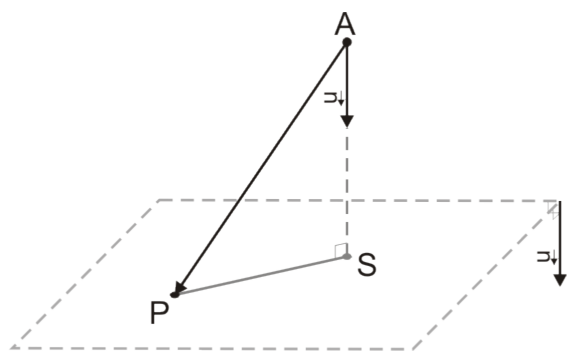
1 Distance between a point and a plane

Given we have

- Point P, lying on a plane
- Unit vector $\vec{n}$, normal to the plane
- Point A



Suppose point S lies on the plane, so that AS is perpendicular to the plane. Then the length of AS is the distance between the point and the plane.

AS is collinear with $\vec{n}$ and $|\vec{n}| = 1$, so

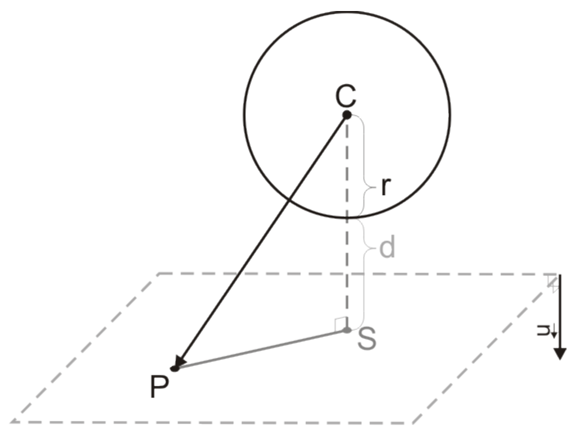
$$AS = \vec{AP} \cdot \vec{n}$$

Note that the plane normal $\vec{n}$ may point towards the half space, in which A is located, or towards the other half space. In the first case, the value for AS will be negative, and in the second it will be positive.
It is so, because fundamentally, rather than length, the value represents the location of S along a ray with initial point A and direction $\vec{n}$. Just as with an axis of a coordinate system, all points "behind" the initial point have negative locations and all "ahead" of the initial point have positive locations.

2 Distance between a sphere and a plane

Given we have

- Point P, lying on a plane
- Unit vector $\vec{n}$, normal to the plane
- Sphere center point C
- Sphere radius r



As described in "Distance between a point and a plane", we can find the distance between the sphere's center and the plane as

$$CS = \vec{RC} \cdot \vec{r}$$

We can distinct between two aspects of what the distance between the sphere and the plane may mean

- As the distance between the sphere volume and the half space volume, which is behind the plane

  $$D = CS - r$$

  This can be interpreted as the distance between the volumes of the sphere and the half space, which has the plane as its boundary and has the plane normal pointing towards it. This value has positive sign only when the sphere is fully outside this half space, without touching the plane.
  If $D \leq -2r$, then the sphere is fully on the other side of the plane.

- As the distance between the sphere's surface and the plane's surface

  $$d = |CS| - r$$

  This is the shortest distance between the two surfaces and as such has positive sign whenever the two don't intersect and negative when they do, disregarding on which side of the plane is the sphere.

3 Distance between ray and perpendicular plane

Often, for bill boarding techniques in graphics, is required to get the distance between a ray and a plane, which is always facing the ray at right angle. Because such plane's orientation is practically defined by the ray's orientation, the plane is defined only by a point, lying on it.
Given we have

- Point P, lying on a plane
- Ray initial point R
- Ray direction, given by the unit vector $\vec{r}$

Because the plane is always oriented towards the ray, its normal vector would be exactly $\vec{r}$ or its reverse.
In this sense, finding the distance is reduced to task of finding the distance between a point and a plane, where R is the point and $\vec{r}$ is the plane's normal vector. This way, the distance d is

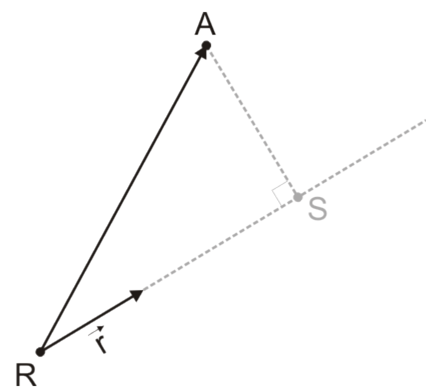
$$d = \vec{RP} \cdot \vec{r}$$

Note that d has positive sign, if the plane is "ahead" of the ray and negative sign, if the plane is "behind" the ray.

4 Distance between a point and a ray

Given we have

- Ray initial point R
- Ray direction, given by the unit vector $\vec{r}$
- Point A



Suppose point S lies on the ray, so that AS is perpendicular to the ray. Then the length of $\vec{AS}$ is the distance between the point and the ray.

4.1 Via dot product

$RS = \vec{RA} \cdot \vec{r}$, because $|\vec{r}| = 1$
$\vec{RS} = \vec{r}RS$, and $\vec{AS} = \vec{RS} - \vec{RA}$, therefore

$$\vec{AS} = \left(\vec{RA} \cdot \vec{r}\right) \vec{r} - \vec{RA}$$

The distance between the point and the ray is simply $\left|\vec{AS}\right|$.

Note that RS is the location of S along the ray, and that it is positive if S is "ahead" of R, negative if S is "behind" R, and zero if $S \equiv R$.

4.2 Via cross product

Because $AS = RA \sin(\angle ARS)$ and $|\vec{r}| = 1$, the distance is simply

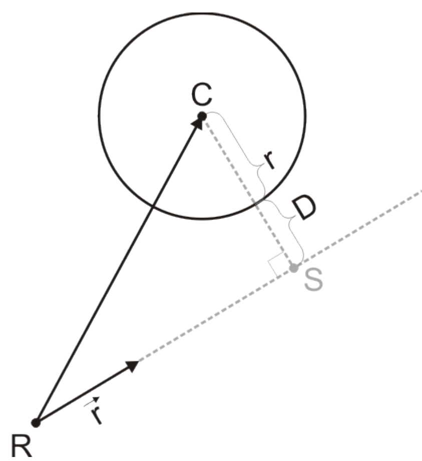
$$|\vec{RA} \times \vec{r}|$$

This approach is more elegant, however if we are interested in the location of S along the ray, we'd need to use dot products anyway.

5 Distance between a sphere and a ray

Given we have

- Ray initial point R
- Ray direction, given by the unit vector $\vec{r}$
- Sphere center point C
- Sphere radius r



As described in "Distance between a point and a ray", we can find the distance between the sphere's center and the ray as

$$CS = \left| \left(\vec{RC} \cdot \vec{r} \right) \vec{r} - \vec{RC} \right| = \\ = |\vec{RC} \times \vec{r}|$$

And we have the distance between the sphere and the ray by simply subtracting the sphere's radius out of d

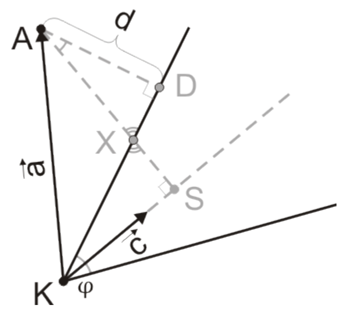
$$D = CS - r$$

Note that if $D < 0$, then the sphere and the plane intersect.

6 Distance between point and a cone

Given we have

- Cone initial point K
- Cone orientation, given by the vector $\vec{c}$
- Cone opening angle φ
- Point A



Let point S lie along $\vec{c}$, so that $AS \perp \vec{c}$
Let AS intersect the cone surface at point X
Let point D lie on the surface of the cone, so that AD is perpendicular to the surface of the cone and is the distance between the point and the cone.

For convenience, substitute $\vec{a} = \vec{KA}$ and $d = AD$.

From $\triangle XAD \rightarrow d = AX \cos(\angle XAD) = AX \cos\varphi$

$AX = AS - XS$

As in "Distance between a point and a ray", we have
$KS = \vec{a} \cdot \vec{c}$
$\vec{AS} = (\vec{a} \cdot \vec{c})\vec{c} - \vec{a} = |\vec{a} \times \vec{c}|$

From $\triangle SKX \rightarrow XS = KX \sin\varphi$
$KS = KX \cos\varphi \rightarrow KX = \frac{KS}{\cos\varphi}$, therefore $XS = \frac{(\vec{a}\cdot\vec{c})\sin\varphi}{\cos\varphi}$

Having calculated AS and XS, we can finally express d

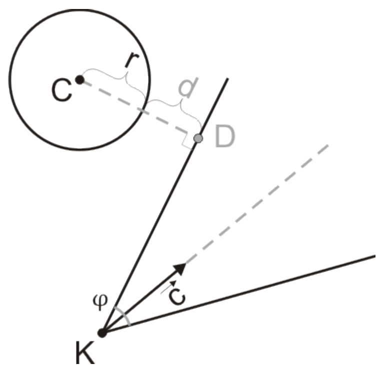
$$\begin{aligned} d &= |(\vec{a} \cdot \vec{c})\vec{c} - \vec{a}| \cos\varphi - (\vec{a} \cdot \vec{c}) \sin\varphi = \\ &= |\vec{a} \times \vec{c}| \cos\varphi - (\vec{a} \cdot \vec{c}) \sin\varphi \end{aligned}$$

Note that if A is inside the cone, the result value will be negative and if the location of A is such that S is "behind" K, then $\vec{a} \cdot \vec{c}$ will be negative.

7 Distance between a sphere and a cone

Given we have

- Cone initial point K
- Cone orientation, given by the vector $\vec{c}$
- Cone opening angle φ
- Sphere center C
- Sphere radius r



As described in "Distance between a point and a cone", we can find the distance between the sphere's center and the cone as

$$CD = |\vec{a} \times \vec{c}| \cos\varphi - (\vec{a} \cdot \vec{c}) \sin\varphi$$

This is the distance between the point C and the cone.
Because, unlike a point, a sphere is a volumetric body, we can take into consideration two aspects of the distance between it and the cone

- The distance between the volumes of the two bodies

  $$d = CD - r$$

  This value has positive sign whenever the sphere is fully outside the cone and negative whenever the volumes of the two bodies intersect

- The distance between the surfaces of the two bodies

  $$D = |CD| - r$$

  This is the shortest distance between the surfaces of the two objects and so it has positive sign whenever the sphere is fully outside or fully inside the cone and negative whenever the surfaces of the two bodies intersect.